

Shell-like Structures in an expanding Quark–Antiquark Plasma

Carsten Greiner^{1†} and Dirk-Hermann Rischke^{2‡},

¹ Institut für Theoretische Physik, Universität Giessen,
D-35392 Giessen, Germany,

² Physics Department, Pupin Physics Laboratories,
Columbia University, NY 10027, USA.

April 1996

Abstract

The particle density distribution emerging from the solution of the Vlasov equation for relativistic, non-interacting particles with spherically symmetric initial conditions is shown to exhibit a shell-like structure for late fixed times in the center-of-mass (CM) frame of the system. A similar phenomenon was recently observed employing the test-particle method to solve the Vlasov equation for quarks with Nambu–Jona-Lasinio–type interactions, and was attributed to the attractive forces among the particles. Contrary to this claim, it is demonstrated here that this effect is of purely relativistic origin and is sensitive only to the mass of the particles and their initial phase-space distribution.

[†]e-mail address: greiner@theorie.physik.uni-giessen.de

[‡]e-mail address: drischke@nt1.phys.columbia.edu

One of the primary goals in present relativistic heavy-ion collision experiments at the AGS at Brookhaven, the SPS at CERN, and in future experiments at Brookhaven's Relativistic Heavy Ion Collider (RHIC) and CERN's Large Hadron Collider (LHC) is the temporary formation and subsequent observation of the so-called quark-gluon plasma (QGP), the deconfined, chirally restored phase of strongly interacting matter [1, 2]. In the common picture of chiral symmetry restoration [2], at sufficiently high temperatures or densities the massive and confined quasi-particle excitations, the constituent quarks, become bare, undressed quarks with current quark masses being much smaller than the constituent quark masses.

Recently, Abada and Aichelin [3] studied the dynamical evolution of a collision-free quark-antiquark system by solving the relativistic Vlasov equation for the one-particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$,

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p} c^2}{E} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} E \cdot \nabla_{\mathbf{p}} f = 0 , \quad (1)$$

where the effective momentum-dependent force

$$-\nabla_{\mathbf{x}} E = -c \nabla_{\mathbf{x}} [\mathbf{p}^2 + M_c^2(\mathbf{x}, t) c^2]^{1/2} = -\frac{M_c(\mathbf{x}, t) c^2}{E} c^2 \nabla_{\mathbf{x}} M_c(\mathbf{x}, t) \quad (2)$$

is generated by a self-consistent gap equation for the constituent quark mass M_c derived within the chiral quark model of Nambu and Jona-Lasinio [4, 5]. The initial condition was a sphere of radius R , homogeneously filled with quarks and antiquarks having a thermal momentum distribution corresponding to a temperature T well above the transition temperature $T_c \sim 140$ MeV. Such “hot spots”, i.e., small-scale fluctuations with sufficiently high (and thermalized) energy density, are expected to occur in present heavy-ion experiments at the SPS [6] and also in future collider experiments at RHIC [7].

The striking observation of Abada and Aichelin was that, in the evolution of the system, most particles remain concentrated in an expanding shell, and that consequently the phase transition takes place at the inner and outer surface of that shell. Thus, the situation is fundamentally different from the naive expectation that the initial fireball expands by retaining its shape and consequently hadronizes from its surface only. For the interpretation of the results, Abada and Aichelin argued as follows: in the non-relativistic case the thermal velocity distribution of particles peaks around $\langle v \rangle = \sqrt{2T/m}$, and in an interaction-free expansion, all the many particles with that velocity will stay together, leading to a corresponding peak in the density, i.e., the observed shell-like structure. For interacting quarks, the attractive forces in the Nambu-Jona-Lasinio model will enhance this effect by trying to keep the (bare) quarks in the region of high density. This conclusion was further motivated by a similar observation found in non-relativistic simulations of expanding nuclear matter, where a shell structure forms after the nucleons have frozen out and enter the liquid-gas transition regime [8].

In contrast, in this note we show that the observation by Abada and Aichelin [3], i.e., the shell-like structure in the density distribution, originates simply from ultrarelativistic kinematics. Moreover, contrary to their line of arguments, we show that in the non-relativistic limit, the density distribution does *not* exhibit such a structure. We will not justify the validity of the Vlasov equation for modelling the expansion of a quark-antiquark plasma.

Obviously, collisions among the constituents are important to achieve the assumed, thermally equilibrated initial condition in the first place. The relevant quantity for thermalization is given by the transport rate Γ_q^{trans} of the quarks in a QGP [9]. Estimates with a simple infrared cutoff suggest a rather high rate of $\Gamma_q^{trans} \approx 0.3 - 0.5 T$ (T denotes the temperature of the plasma), however, a more detailed treatment of the infrared sector in those rates by means of resummation methods raises the question whether one can trust those perturbative calculations at all [9].

The collision rate may be high enough to maintain (local) thermal equilibrium even during the expansion. In that case the system's evolution is governed by ideal hydrodynamics. We shall compare the relativistic hydrodynamical solution for an expanding sphere with the results obtained from solving the Vlasov equation.

On the other hand, it was also recently speculated that, in particular shortly before the phase transition happens, the collisions in the plasma become so rare that their effect on the overall dynamics can be neglected. Then, the system freezes out already before the transition occurs and it is appropriate to model the non-thermal dynamics with the Vlasov equation [10].

There is a very simple, intuitive explanation why a shell occurs in the relativistic expansion of the fireball: expanding matter piles up in a shell, because the velocity of light c limits the maximum particle velocity. Suppose the particles are ultrarelativistic, i.e., massless. In that case, the velocity of *all* particles equals the velocity of light, $v \equiv c$. Obviously, without collisions, at times $t \gg R/c$ they can only be located in a spherical shell of thickness $2R$, moving outward with light velocity. (The particle density in that shell is not necessarily constant, but depends on the initial distribution of momenta and coordinates in the sphere, see below). The interior of the shell will be completely void of particles. For particles with finite mass, the thickness of the shell is roughly

$$d \approx 2R + \Delta v t ,$$

where $\Delta v \equiv |\langle v^2 \rangle - \langle v \rangle^2|^{1/2}$ characterizes the average spread of the velocity distribution. That spread grows with increasing mass of the particles, such that in the non-relativistic limit the interior of the shell is filled with particles, thus restoring the naive expectation for the expansion of a sphere (and disproving the arguments presented in [3]).

As will be argued below, the modification of the momenta and velocities of the particles due to the force term (2) and the time dependence of the mass are small, so that the generic features of phase-space dynamics will be represented by the free-streaming limit of eq. (1),

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f = 0 , \quad (3)$$

where $\mathbf{v} \equiv \mathbf{p}c^2/E$ and the mass M_c of the quarks is assumed to be independent of time. In the following we explicitly construct the solution for the particle density distribution¹

$$\rho(\mathbf{x}, t) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t) . \quad (4)$$

¹We choose units $\hbar = k_B = 1$.

Our initial condition will ultimately be the same as in [3], i.e., the particles are thermally distributed in momentum space and homogeneously distributed in coordinate space within a sphere of radius R . For the moment, however, arbitrary (though homogeneously filled) initial coordinate-space volumina V and arbitrary initial momentum distributions $n(\mathbf{p})$ are allowed.

We introduce the characteristic function of the initial volume, $\Theta_V(\mathbf{x}) = 1$ for $\mathbf{x} \in V$, and 0 elsewhere. Then, for a coordinate-space distribution which is homogeneous in V , the initial phase-space distribution separates as

$$\begin{aligned} f(\mathbf{x}, \mathbf{p}, 0) &= \Theta_V(\mathbf{x}) n(\mathbf{p}) \\ &\equiv \int d^3\mathbf{a} \Theta_V(\mathbf{a}) f_{\mathbf{a}}(\mathbf{x}, \mathbf{p}, 0) \end{aligned} \quad (5)$$

where we defined the phase-space distribution for a δ -like source,

$$f_{\mathbf{a}}(\mathbf{x}, \mathbf{p}, 0) = \delta^3(\mathbf{x} - \mathbf{a}) n(\mathbf{p}) . \quad (6)$$

Since (3) is a homogeneous linear differential equation, the solution $f(\mathbf{x}, \mathbf{p}, t)$ has the form

$$f(\mathbf{x}, \mathbf{p}, t) = \int d^3\mathbf{a} \Theta_V(\mathbf{a}) f_{\mathbf{a}}(\mathbf{x}, \mathbf{p}, t) . \quad (7)$$

The solutions $f_{\mathbf{a}}$ to eq. (3) (for time-independent particle mass) are readily obtained:

$$f_{\mathbf{a}}(\mathbf{x}, \mathbf{p}, t) = f_{\mathbf{a}}(\mathbf{x} - \mathbf{v}t, \mathbf{p}, 0) = \delta^3(\mathbf{x} - \mathbf{v}t - \mathbf{a}) n(\mathbf{p}) . \quad (8)$$

The particle density distribution (4) $\rho_{\mathbf{a}}(\mathbf{x}, t)$ for a δ -like source (6) located at $\mathbf{x} = \mathbf{a}$ follows after a careful evaluation of the δ -function as

$$\begin{aligned} \rho_{\mathbf{a}}(\mathbf{x}, t) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} n(\mathbf{p}) \delta^3\left(\mathbf{x} - \frac{\mathbf{p}c^2}{E}t - \mathbf{a}\right) \\ &= \frac{1}{(2\pi)^3} n(m \cosh \eta \mathbf{u}) \cosh^5 \eta \left(\frac{m}{t}\right)^3 \Theta(1 - \tanh \eta) , \quad t > 0 , \end{aligned} \quad (9)$$

where

$$\mathbf{u} \equiv \frac{\mathbf{x} - \mathbf{a}}{t} , \quad \tanh \eta \equiv |\mathbf{u}|/c . \quad (10)$$

Obviously, $m \cosh \eta \mathbf{u}$ is the momentum and $mc^2 \cosh \eta$ the energy of the particles. The step function assures causal expansion. The non-relativistic limit corresponds to $|\mathbf{u}| \ll c$, $\cosh \eta \approx 1$ and (9) becomes

$$\rho_{\mathbf{a}}^{\text{n.r.}}(\mathbf{x}, t) = \frac{1}{(2\pi)^3} n(m \mathbf{u}) \left(\frac{m}{t}\right)^3 . \quad (11)$$

If the initial momentum distribution $n(\mathbf{p})$ is isotropic, i.e., $n(|\mathbf{p}|)$, and a monotonously decreasing function of $|\mathbf{p}|$, as it is the case for a thermal (Bose–Einstein, Fermi–Dirac, or Maxwell–Boltzmann) distribution, the density distribution $\rho_{\mathbf{a}}(\mathbf{x}, t)$ is also isotropic, $\rho_{\mathbf{a}}(r, t)$, and, in the non-relativistic limit, a decreasing function of the (radial) distance $r \equiv |\mathbf{x} - \mathbf{a}|$ from the source. This disproves the argument presented in [3] about the formation of a shell-like structure in the non-relativistic limit.

This behaviour changes completely in the ultrarelativistic regime due to the additional factor $\cosh^5 \eta$. This term is a monotonously increasing function of r and, in combination with the momentum distribution function, thus leads to a pronounced peak in the density distribution. In Fig. 1 the quantity $(ct)^3 \rho_{\mathbf{a}}(r, t)$ is plotted as a function of r/ct for various masses m . (Note that this representation is time-invariant.) A Fermi distribution

$$n(|\mathbf{p}|) \equiv n_F(|\mathbf{p}|) = \frac{1}{e^{E/T} + 1}, \quad E \equiv c \sqrt{\mathbf{p}^2 + (mc)^2},$$

was chosen with a temperature $T = 160$ MeV, slightly larger than the expected critical temperature T_c . The shape of the distribution is very sensitive to the ratio mc^2/T . One clearly sees how the peak structure emerges already when mc^2/T is of order one, as most of the particles are already moving with a velocity close to c . (If $mc^2/T \ll 1$ the width of the velocity distribution is proportional to $\sim (mc^2/T)^2$.)

For a uniformly filled sphere of radius R ,

$$\Theta_V(\mathbf{a}) = \begin{cases} 1 & \text{for } |\mathbf{a}| \leq R \\ 0 & \text{for } |\mathbf{a}| > R \end{cases}.$$

The particle density distribution $\rho(\mathbf{x}, t)$ is then obtained as

$$\rho(\mathbf{x}, t) = \int d^3\mathbf{a} \Theta_V(\mathbf{a}) \rho_{\mathbf{a}}(\mathbf{x}, t). \quad (12)$$

In principle, one should multiply (12) by the degeneracy $d_q = 12$ of light quarks (and similarly for the antiquarks), but this factor is, for the sake of simplicity, omitted in the following. Eq. (12) can be reduced to a one-dimensional integral and takes the form

$$\begin{aligned} \rho(\tilde{r}, \tilde{t}) &= \left(\frac{mc}{2\pi\tilde{t}}\right)^3 \frac{\pi}{\tilde{r}} \Theta(\tilde{t} - \tilde{r} + 1) \int_{\tilde{r}-1}^{\min(\tilde{r}+1; \tilde{t})} dz z \left(1 - (z - \tilde{r})^2\right) n(mc \sinh \eta_z) \cosh^5 \eta_z \\ &\text{for } \tilde{r} > 1; \end{aligned} \quad (13)$$

$$\begin{aligned} \rho(\tilde{r}, \tilde{t}) &= \left(\frac{mc}{2\pi\tilde{t}}\right)^3 2\pi \left[2 \int_0^{\min(1-\tilde{r}; \tilde{t})} dz z^2 n(mc \sinh \eta_z) \cosh^5 \eta_z \right. \\ &\quad \left. + \frac{1}{2\tilde{r}} \Theta(\tilde{t} + \tilde{r} - 1) \int_{1-\tilde{r}}^{\min(1+\tilde{r}; \tilde{t})} dz z \left(1 - (z - \tilde{r})^2\right) n(mc \sinh \eta_z) \cosh^5 \eta_z \right] \\ &\text{for } \tilde{r} < 1, \end{aligned}$$

where the dimensionless variables $\tilde{r} = r/R$ and $\tilde{t} = ct/R$ have been introduced and $\tanh \eta_z \equiv z/\tilde{t}$. Expression (13) has to be integrated numerically.

In Fig. 2 the resulting density profile $\rho(\tilde{r}, \tilde{t})$ is shown for different times \tilde{t} between 0.1 and 3. Again, the initial momentum distribution is a Fermi distribution n_F with a temperature $T = 160$ MeV. The mass of the quarks is taken as $m(\equiv M_c) = 50 \text{ MeV}/c^2$. For times $\tilde{t} < 1$

the density in the central region $\tilde{r} \leq 1 - \tilde{t}$ remains constant due to causality. In other words, it remains constant because as many particles leave an infinitesimal volume element as do enter. However, since most particles move with nearly light velocity (the thermal velocity is $\langle v \rangle \approx 0.9858 c$ for the present situation, while the spread of the velocity distribution is only $\Delta v \equiv |\langle v^2 \rangle - \langle v \rangle^2|^{1/2} \approx 0.0031 c$), for times $\tilde{t} > 1$ no particles can enter the central region around $\tilde{r} = 0$ any more, so that it becomes completely depleted. Thus, a shell starts to move outwards at time $\tilde{t} = 1$, as one clearly recognizes in Fig. 2. As expected, that shell has a width of $2 R$ and travels with the velocity of light.

In Fig. 3 density profiles $\rho(\tilde{r}, 1.5)$ are shown for different masses. For all masses smaller than $300 \text{ MeV}/c^2$ relativistic effects become important, and a shell structure emerges in the temporal evolution.

This behaviour is quite similar to the relativistic, ideal hydrodynamical expansion of a spherical fireball [11]. Also in this case, the particle density distribution exhibits a shell structure at fixed times in the CM frame of the fireball. The reason is that moving matter at finite r experiences relativistic time dilation [12] with respect to matter in the center (which is at rest due to symmetry) and thus dilutes less rapidly. To confirm this, and to investigate the effect of a varying particle mass on the expansion, we solved the equations of ideal relativistic hydrodynamics,

$$\partial_\mu T^{\mu\nu} = 0 \quad (14)$$

($T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu}$ is the energy–momentum tensor of an ideal fluid, ϵ , p are energy density and pressure in the local rest frame of a fluid element, moving with 4–velocity u^μ in the CM frame of the fireball, $g^{\mu\nu} = \text{diag}(+, -, -, -)$ is the metric tensor), for a spherically symmetric initial fireball of radius R and temperature $T = 160 \text{ MeV}$ and for ideal Fermi gas equations of state

$$\epsilon(T) = \frac{1}{2\pi^2} \int_0^\infty dp p^2 \sqrt{p^2 + m^2 c^2} n_F(p) , \quad (15)$$

$$p(T) = \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^4}{\sqrt{p^2 + m^2 c^2}} n_F(p) , \quad (16)$$

with masses m varying between 10 and $940 \text{ MeV}/c^2$. The corresponding particle densities are

$$\rho(T) = \frac{1}{2\pi^2} \int_0^\infty dp p^2 n_F(p) . \quad (17)$$

The hydrodynamical equations (14) are solved with the (one–dimensional) relativistic Harten–Lax–van Leer–Einfeldt algorithm [12], supplemented with a Sod predictor–corrector step [13] to account for spherical geometry [14].

In Fig. 4 we show the corresponding density profiles at time $\tilde{t} = 2.97$. The situation is qualitatively very similar to Fig. 3, which is remarkable considering the fact that ideal hydrodynamics corresponds to the limit of an infinite collision rate (and thus immediate local thermodynamical equilibrium). The main quantitative differences are (a) the time \tilde{t} has to be about twice as large as before to obtain density distributions similar to Fig. 3, and (b) the quark densities are smaller. The reason for (a) is that for a time $\tilde{t} = 1.5$ the hydrodynamical

rarefaction has not yet reached the center of the fireball (the rarefaction velocity is the velocity of sound $c_s \leq c/\sqrt{3} \approx 0.5774c$, in contrast to the free-streaming case where the rarefaction wave travels with the velocity of light c), and thus the shell structure could not possibly have formed. The reason for (b) is that the quark density is not conserved in the solution of (14), in contrast to the conservation of the quark number in the free-streaming scenario: due to the work performed in the hydrodynamical expansion the system cools, and thus temperature and, consequently, particle density decrease. Thus, the densities are smaller in Fig. 4 than in Fig. 3. In order to conserve the quark number in the hydrodynamical evolution, one would have to supplement the equations of motion (14) with the continuity equation for the quark number current, but this would also require introducing a finite quark chemical potential in the equation of state. This is certainly beyond the scope of our present work, where we are interested in qualitative similarities between relativistic free-streaming and hydrodynamical solutions.

Finally, we address the potential influence of the self-consistent mass term and the corresponding force (2) on the expansion dynamics. The mass $M_c(r, t)$ influences the velocities of the particles in two ways. First, due to its spatial dependence a force is generated which changes the momentum of a particle in time according to

$$\dot{\mathbf{p}} = \dot{p} \mathbf{e}_{\mathbf{p}} + p \dot{\mathbf{e}}_{\mathbf{p}} = -\frac{M_c(r, t)c^2}{E} c^2 \nabla_x M_c(r, t) . \quad (18)$$

Second, the velocity of a particle changes because of the time dependence of the mass; in particular it will continuously decrease if $M_c(t)$ increases with time as in the present case.

According to [3] the initial mass of the particles is $M_c \approx 50 \text{ MeV}/c^2$. For a temperature of $T \approx 240 \text{ MeV}$ (used in [3]) the thermal velocity is $\sim 0.9931c$. To estimate the typical change in momentum, the modulus of the gradient in the mass M_c is (conservatively) read off the results of [3] as $|\nabla_x M_c(r, t)c^2| \leq 500 \text{ MeV/fm}$, acting over a period in time $\Delta t \approx 0.5 \text{ fm}/c$. Taking $\langle E \rangle$ and $\langle pc \rangle \approx 3T$ and an average mass $\langle M_c c^2 \rangle \approx 100 \text{ MeV}$ the change in the (average) momentum is estimated as

$$\begin{aligned} \frac{\langle \Delta p \rangle}{\langle p \rangle} &= \frac{\langle \dot{p} \rangle \Delta t}{\langle p \rangle} \leq \frac{1}{\langle p \rangle} \frac{\langle M_c c^2 \rangle}{\langle E \rangle} |\nabla_x M_c(r, t)c^2| \Delta t \approx 0.05 \\ \langle \Delta \mathbf{e}_{\mathbf{p}} \rangle &= \langle \dot{\mathbf{e}}_{\mathbf{p}} \rangle \Delta t \leq \frac{1}{\langle p \rangle} \frac{\langle M_c c^2 \rangle}{\langle E \rangle} |\nabla_x M_c(r, t)c^2| \Delta t \approx 0.05 . \end{aligned} \quad (19)$$

Hence, the typical momentum of a particle is to good approximation conserved both in magnitude and direction; the change is at most a few percent.

For approximate momentum conservation, $p_i(t = t_i)c \approx 3T \approx p_f(t = t_f)c$, the change in the typical velocity of the particles due to the increase of the mass from $M_c(t_i) \approx 50 \text{ MeV}/c^2$ to $M_c(t_f) \approx 300 \text{ MeV}/c^2$ is readily obtained as

$$v_{i; M_c=50 \text{ MeV}/c^2} - v_{f; M_c=300 \text{ MeV}/c^2} \approx 0.09c . \quad (20)$$

All particles with large (thermal) velocities will be decelerated by typically ten percent. As a consequence, the formation of the shell will be somewhat delayed. Furthermore, this deceleration of the particles will stabilize the region of expanding matter, where the mass of

the particles is still rather low, i.e., the interior of the outgoing shell. In conclusion, taking a self-consistent mass into account will only have minor effects on the dynamics. The generic structure of phase-space dynamics is dominated by relativistic kinematics as given by the solution of (3).

To conclude, in this note we have investigated the free-streaming expansion of a fireball consisting of quarks and/or antiquarks. We explicitly constructed the solution for spherically symmetric initial conditions and confirmed the existence of a shell-like structure in the expansion at late times [3]. Contrary to the arguments of [3], we have explained that structure as arising solely from (and only for) relativistic kinematics. Interestingly enough, similar structures occur in the ideal hydrodynamical limit (where interaction rates are infinite), which again confirms that their origin is kinematical rather than due to specific details of the interaction between the constituent particles.

The overall decrease in the density of the shell is $\sim 1/t^3$, cf. eq. (13). If the critical density for the phase transition is reached when the shell was already formed (for times $t \geq R/c$, cf. Fig. 2), the transition necessarily happens at the outer as well as the inner surface of the shell. This is what has been observed in [3]. In principle, however, it is also conceivable that the phase transition happens before the shell structure has formed, for instance when the initial temperature is only slightly above the critical temperature T_c . Then the fireball will hadronize only in the surface region where the density decreases fast at times $t < R/c$.

The analytic solutions presented here may also serve as benchmark tests for numerical algorithms like the test-particle method. Another test case with a solution that can be readily obtained by similar means (but will not be presented here for the sake of brevity) is the free-streaming expansion of a slab (finite thickness in one dimension, but infinite extension in the other two). In that case, no shell structure emerges (because sufficiently many particles enter the central region from other parts of the slab). This is in contrast to the analogous hydrodynamical expansion problem [12].

Acknowledgements:

Discussions with J. Aichelin are acknowledged. Both authors thank the Alexander von Humboldt Stiftung for partial support under the Feodor Lynen program. C.G. acknowledges support by the BMBF and GSI Darmstadt, D.H.R. acknowledges support by the Director, Office of Energy Research, Division of Nuclear Physics of the Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract No. DE-FG-02-93ER-40764.

References

- [1] *Quark Matter '95*, Nucl. Phys. **A590**, 1 (1995), and earlier *Quark Matter* proceedings.
- [2] E.V. Shuryak, Phys. Rep. **61**, 72 (1980); **115**, 151 (1984);
D. Gross, R. Pisarski, and L. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
- [3] A. Abada and J. Aichelin, Phys. Rev. Lett. **74**, 3130 (1995).
- [4] for a review, see: U. Vogl and W. Weise, Prog. Part. Nucl. Phys. **27**, 195 (1991);
S.P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992).
- [5] W.M. Zhang and L. Wilets, Phys. Rev. **C45**, 1900 (1992).
- [6] J. Aichelin and K. Werner, Phys. Lett. **B300**, 158 (1993);
K. Werner, Phys. Rev. Lett. **73**, 1594 (1994).
- [7] M. Gyulassy, Proceedings of the International Conference on Nuclear Physics at the Turn of the Millennium: Structure of Vacuum and Elementary Matter (World Scientific, 1996);
M. Gyulassy, D.H. Rischke, and B. Zhang, Columbia University preprint CU-TP-707.
- [8] D.H.E. Gross, B.-A. Li, and A.R. De Angelis, Ann. Phys. **1**, 467 (1992);
W. Bauer, G. Bertsch, and H. Schulz, Phys. Rev. Lett. **69**, 1888 (1992).
- [9] M.H. Thoma, Phys. Rev. **D49**, 451 (1994).
- [10] L.P. Csernai and I.N. Mishustin, Phys. Rev. Lett. **74**, 5005 (1995).
- [11] G.S. Bisnovatyi-Kogan and M.V.A. Murzina, Phys. Rev. **D52**, 4380 (1995).
- [12] D.H. Rischke, S. Bernard, and J.A. Maruhn, Nucl. Phys. **A595**, 346 (1995).
- [13] G.A. Sod, J. Fluid Mech. **83**, 785 (1977).
- [14] D.H. Rischke and M. Gyulassy, in preparation.

Figure captions:

Figure 1:

The density profile $(ct)^3 \rho_{\mathbf{a}}(r, t)$ of a point source for various masses m as a function of r/ct . The initial momentum distribution is a Fermi distribution with temperature $T = 160$ MeV.

Figure 2:

The density profile $\rho(\tilde{r}, \tilde{t})$ at various stages \tilde{t} of the free-streaming expansion of a fireball. The initial momentum distribution is a Fermi distribution with $T = 160$ MeV. The mass of the particles is taken as $m = 50$ MeV/ c^2 . At times $\tilde{t} > 1$ a shell structure emerges in the radial expansion.

Figure 3:

The density profile $\rho(\tilde{r}, \tilde{t})$ at a fixed time $\tilde{t} = 1.5$ for the free-streaming expansion of a fireball. The mass of the particles is varied between 10 and 940 MeV/ c^2 to demonstrate the relativistic effects in the expansion.

Figure 4:

The density profile $\rho(\tilde{r}, \tilde{t})$ at a fixed time $\tilde{t} = 2.97$ for the ideal relativistic hydrodynamical expansion of a fireball. The equation of state is that of an ideal gas of relativistic particles with mass varying between 10 and 940 MeV/ c^2 .













